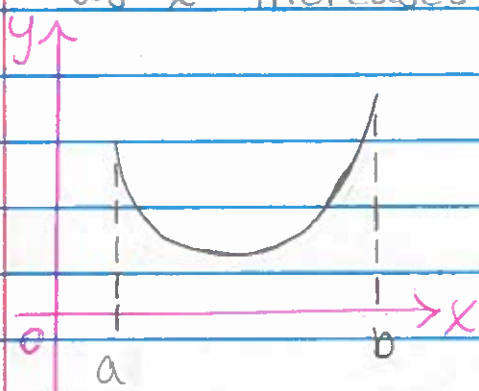


The Second Derivative Test

Concavity

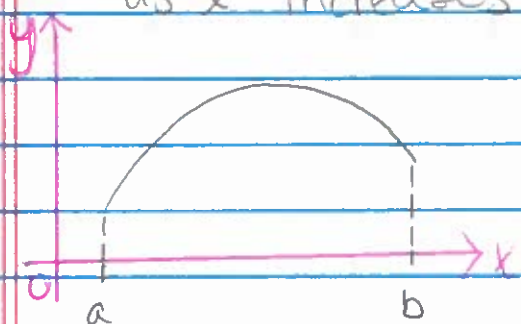
- i) a function $y = f(x)$ is said to be **concave up** on $a \leq x \leq b$ if its first derivative increases as x increases on that interval.



Slopes of subsequent tangent lines increase in value as x increases.

A concave up curve is part of a "smiley".

- ii) A function $y = f(x)$ is said to be **concave down** on $a \leq x \leq b$ if its first derivative decreases as x increases on that interval.



slopes of subsequent tangent lines decrease in value as x increases.

A concave down curve is part of a "frowney".

Review

Slopes of tangent lines are calculated by using the first derivative $\frac{dy}{dx} = f'(x)$

In thinking...

↳ If $y = f(x)$ increases as x increases,

⇒ $\frac{dy}{dx} = f'(x)$ is positive on that interval.

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On a concave up curve,

$\frac{dy}{dx} = f'(x)$ increases as x increases.

$\Rightarrow \frac{d^2y}{dx^2} = f''(x)$ is positive on that interval.

$\hookrightarrow$ If $y=f(x)$ decreases as x increases,

$\Rightarrow \frac{dy}{dx} = f'(x)$ is negative on that interval.

On a concave down curve,

$\frac{dy}{dx} = f'(x)$ decreases as x increases.

$\Rightarrow \frac{d^2y}{dx^2} = f''(x)$ is negative on that interval.

Notation

Let us say we have an original function $y=f(x)$

The first derivative is

$$\frac{dy}{dx} = f'(x)$$

The derivative of the first derivative is the second derivative:

$$\frac{d^2y}{dx^2} = f''(x)$$

The derivative of the second derivative is the third derivative

$$\frac{d^3y}{dx^3} = f'''(x)$$

The derivative of the third derivative is the fourth derivative

$$\frac{d^4y}{dx^4} = f^{(4)}(x)$$

and so on. The general n^{th} derivative is:

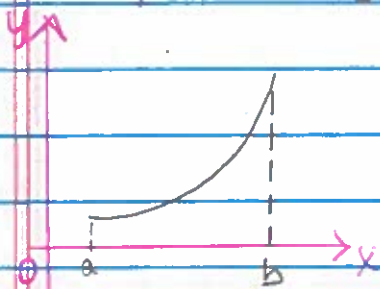
$$\boxed{\frac{d^ny}{dx^n} = f^{(n)}(x)}$$

Ans. 2p3 MA121

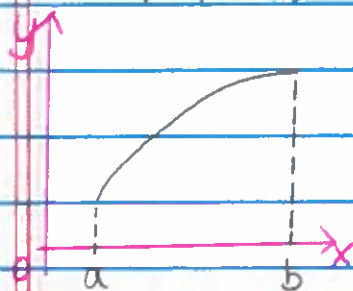
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Possibilities

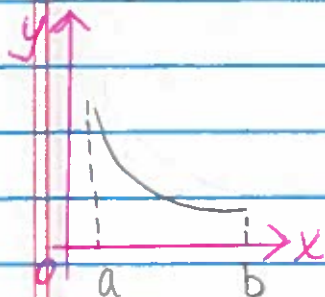
i) $y = f(x)$ is increasing and concave up on $a \leq x \leq b$



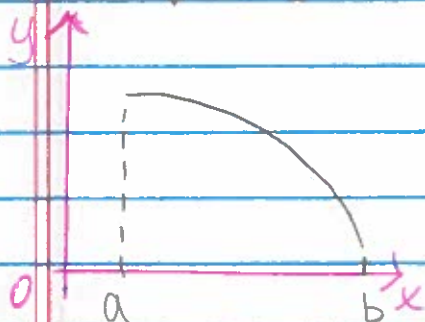
ii) $y = f(x)$ is increasing and concave down on $a \leq x \leq b$



iii) $y = f(x)$ is decreasing and concave up on $a \leq x \leq b$



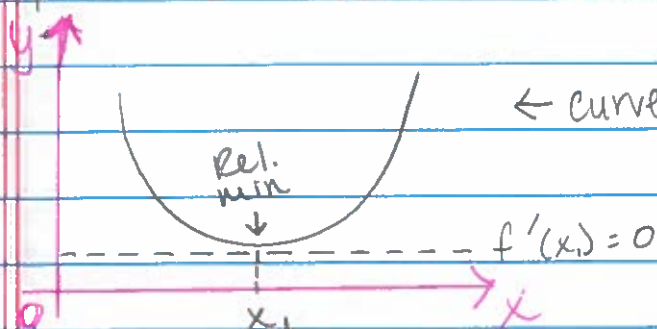
iv) $y = f(x)$ is decreasing and concave down on $a \leq x \leq b$



... and combinations of the above.

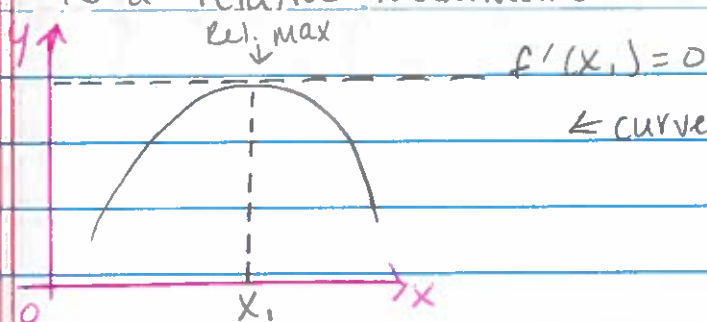
The Second Derivative test

i) if $f'(x_1) = 0$ and $f''(x_1) = \text{positive}$, this critical point is a relative minimum.



← curve is concave up ie. $f''(x_1) = \text{positive}$

ii) If $f'(x_1) = 0$ and $f''(x_1) = \text{negative}$, this critical point is a relative maximum.

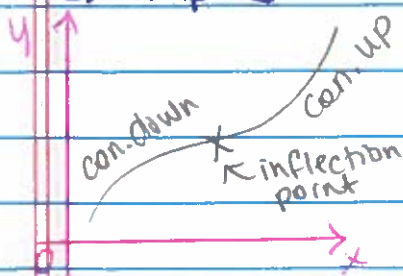


← curve is concave down ie. $f''(x_1) = \text{neg}$

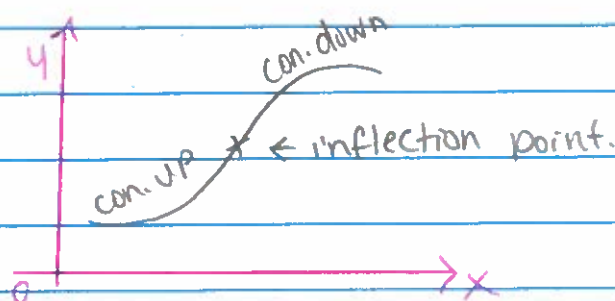
iii) If $f'(x_1) = 0$ and $f''(x_1) = 0$ (does not exist), then the second derivative test fails. This critical point may be a relative maximum or a relative minimum or an inflection point.

Inflection points are where the function changes concavity provided these points are points of continuity.

Examples



OR



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Inflection points may occur when:

$$\hookrightarrow \frac{d^2y}{dx^2} = f''(x_1) = 0 \quad \text{OR}$$

$$\hookrightarrow \frac{d^2y}{dx^2} = f''(x_1) \text{ does not exist.}$$

However, this is a one-way street IF:

$$\hookrightarrow \frac{d^2y}{dx^2} = f''(x_1) = 0 \quad \text{OR}$$

$$\hookrightarrow \frac{d^2y}{dx^2} = f''(x_1) \text{ does not exist.}$$

This point, $x = x_1$, may or may not be an inflection point.

Ex.

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Classify the critical points

$$f(x) = x^3 \ln(x) \quad \text{Domain: } x > 0$$

To find critical points, we need $f'(x)$

$$\text{Given } f(x) = x^3 \ln x$$

$$\Rightarrow f'(x) = (\ln x)(x^3)' + (x^3)(\ln x)'$$

$$\Rightarrow f'(x) = (\ln x)(3x^2) + (x^3)(1/x)$$

$$\Rightarrow f'(x) = 3x^2 \ln x + x^2$$

$$\Rightarrow f'(x) = x^2(3 \ln x + 1)$$

Critical Points occur when

$$(i) f'(x) = 0$$

$$\Rightarrow x^2(3 \ln x + 1) = 0$$

$$\Rightarrow x^2 = 0 \quad \text{or} \quad 3 \ln x + 1 = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad \ln x = -1/3$$

$$\Rightarrow x = 0 \quad \text{or} \quad x = e^{-1/3} \approx 0.7165313 \dots$$

We reject $x = 0$ since it is not on the domain. We can

accept $x = e^{-1/3}$ as it is on the domain.

$$(ii) f'(x) \text{ does not exist, but } f'(x) = x^2(3 \ln x + 1)$$

exists everywhere on the domain so this criteria does not yield any critical point.

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$\therefore f(x) = x^3 \ln x$ has only one critical point, $x = e^{-1/3}$
 Let us use the second derivative test to classify this critical point.

From $f'(x) = x^2(3 \ln x + 1)$

$$\begin{aligned} \Rightarrow f''(x) &= [3 \ln x + 1](x^2)' + (x^2)(3 \ln x + 1)' \\ &= (3 \ln x + 1)(2x) + (x^2)(3/x) \\ &= (3 \ln x + 1)(2x) + 3x \end{aligned}$$

plugging in the critical point, $x = e^{-1/3}$,

$$f''(e^{-1/3}) = (3 \ln(e^{-1/3}) + 1)(2e^{-1/3}) + 3e^{-1/3}$$

$$= (3(-1/3) + 1)(2e^{-1/3}) + 3e^{-1/3}$$

$$\begin{aligned} \Rightarrow f''(e^{-1/3}) &= (-1 + 1)(2e^{-1/3}) + 3e^{-1/3} \\ &= 0 + 3e^{-1/3} \end{aligned}$$

$$\Rightarrow f''(e^{-1/3}) = 3e^{-1/3} > 0$$

$\therefore$ since $f'(e^{-1/3}) = 0$ and $f''(e^{-1/3}) = \text{positive}$,
 $x = e^{-1/3}$ is a relative minimum.

* $e^x = \text{positive}$, $x = \ln(\text{positive})$

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Find intervals of concavity and state any inflection points.

$f(x) = \frac{e^x}{x}$ Domain $x < 0$ or $x > 0$ since $x \neq 0$

Division by zero renders a function undefined.

To find intervals of concavity, we need $f''(x)$, or the second derivative.

Given $f(x) = \frac{e^x}{x}$

$$\begin{aligned} \Rightarrow f'(x) &= (x)(e^x)' - (e^x)(x)' \\ &= \frac{xe^x - e^x(1)}{x^2} \end{aligned}$$

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$$\Rightarrow f'(x) = \frac{xe^x}{x^2} - \frac{e^x}{x^2}$$

$$\Rightarrow f'(x) = \frac{e^x}{x} - \frac{e^x}{x^2}$$

$$\Rightarrow f'(x) = f(x) - \frac{e^x}{x^2}$$

$$\Rightarrow f''(x) = f'(x) - \left(\frac{(x^2)(e^x)' - (e^x)(x^2)'}{(x^2)^2} \right)$$

$$= \frac{e^x}{x} - \frac{e^x}{x^2} - \left(\frac{x^2 e^x - e^x (2x)}{x^4} \right)$$

$$= \frac{e^x}{x} - \frac{e^x}{x^2} - \left(\frac{x^2 e^x}{x^4} - \frac{2xe^x}{x^4} \right)$$

$$= \frac{e^x}{x} - \frac{e^x}{x^2} - \left(\frac{e^x}{x^2} - \frac{2e^x}{x^3} \right)$$

$$= \frac{e^x}{x} - \frac{e^x}{x^2} - \frac{e^x}{x^2} + \frac{2e^x}{x^3}$$

$$\Rightarrow f''(x) = \frac{e^x}{x} - \frac{2e^x}{x^2} + \frac{2e^x}{x^3}$$

$$\Rightarrow f''(x) = \frac{x^2 e^x}{x^3} - \frac{2xe^x}{x^3} + \frac{2e^x}{x^3}$$

$$\Rightarrow f''(x) = \frac{x^2 e^x - 2xe^x + 2e^x}{x^3}$$

$$\Rightarrow f''(x) = \frac{e^x(x^2 - 2x + 2)}{x^3}$$

Inflection Points are where concavity changes. Inflection points may occur where (i) $f''(x) = 0$

$$\Rightarrow \frac{e^x(x^2 - 2x + 2)}{x^3} = 0$$

$$\Rightarrow e^x(x^2 - 2x + 2) = 0$$

$$\Rightarrow x^2 - 2x + 2 = 0 \text{ and } e^x \neq 0$$

2a

$$\Rightarrow x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

This has no solution. so $f''(x) = 0$ does not yield possible inflection points.

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(ii) $f''(x)$ does not exist.

$$\Rightarrow \frac{e^x(x^2 - 2x + 2)}{x^3} \rightarrow \text{DNE}$$

$$\Rightarrow x^3 = 0$$

 $\Rightarrow x = 0$ $\leftarrow$ but this is not on the domain.

This criteria also does not yield possible inflection points. However, we can still determine the concavity of $f(x) = \frac{e^x}{x}$

	$x < 0$	0	$x > 0$
Sign of e^x	positive		positive
Sign of $x^2 - 2x + 2$	positive		positive
Sign of x^3	negative		positive
Sign of $f''(x) = \frac{e^x(x^2 - 2x + 2)}{x^3}$	negative		positive
Concavity of $f(x) = \frac{e^x}{x}$	concave down $\cap$		concave up $\cup$

$f(x) = \frac{e^x}{x}$ is concave up on $x > 0$, concave down on $x < 0$. However, $x = 0$ is not an inflection point, rather, it is a vertical asymptote.

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$f(x) = x \ln x$ Domain is $x > 0$ To find possible inflection points, we need $f''(x)$.

Given $f(x) = x \ln x$,

$$\begin{aligned} \Rightarrow f'(x) &= (\ln x)(x)' + (x)(\ln x)' \\ &= \ln x + 1 + (x)(\frac{1}{x}) \end{aligned}$$

$$\Rightarrow f'(x) = \ln x + 1$$

$$\Rightarrow f''(x) = \frac{1}{x}$$

Inflection points may occur when (i) $f''(x) = 0$

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$\Rightarrow 1/x = 0$, but this is impossible since $1/x \neq 0$

(ii) $f''(x)$ does not exist.

$\Rightarrow 1/x$ DNE

$\Rightarrow x = 0$, but this is not on the domain.

So, $f(x) = x \ln x$ has no inflection points.

sign of $f'(x) = 1/x$ $\begin{array}{c} 0 \\ \bullet \end{array} \xrightarrow[\text{positive}]{x > 0} x$

$\therefore f(x) = x \ln x$ is always concave up on its domain.

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$f(x) = xe^x$ Domain $-\infty < x < \infty$ To find inflection points, we need $f''(x)$

Given $f(x) = xe^x$

$$\begin{aligned} \Rightarrow f'(x) &= (e^x)(x)' + (x)(e^x)' \\ &= e^x + xe^x \end{aligned}$$

$$\Rightarrow f'(x) = e^x + f(x)$$

$$\Rightarrow f''(x) = e^x + f'(x)$$

$$\Rightarrow f''(x) = e^x + e^x + xe^x$$

$$\Rightarrow f''(x) = 2e^x + xe^x$$

Inflection points may occur when (i) $f''(x) = 0$

$$\Rightarrow e^x(2+x) = 0$$

$$\Rightarrow 2+x = 0 \text{ and } e^x \neq 0$$

$\Rightarrow x = -2$ $\leftarrow$ This is acceptable since it is on the domain.

(ii) $f''(x)$ does not exist but $f''(x) = e^x(2+x)$ exists everywhere on the domain

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 $x < -2$ -2 $x > -2$ x

Sign of e^x	positive	positive
Sign of $2+x$	negative	positive
Sign of $f''(x) = e^x(2+x)$	negative	positive
Concavity of $f(x) = xe^x$	concave down	concave up

Inflection
Point

$f(x) = xe^x$ is up on $x > -2$ and concave down on $x < -2$, And $x = -2$ is an inflection point since concavity changes at that point and is on the domain.